

# Short monadic second order sentences about sparse random graphs

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## Abstract

In this paper, we study the so-called zero-one laws for the Erdős–Rényi random graph model  $G(n, p)$  in the case when  $p = n^{-\alpha}$  for  $\alpha > 0$ . For a given class of properties and  $p$ , we say that  $G(n, p)$  obeys the zero-one law if any formula from the class either a.a.s. holds or a.a.s. does not hold in  $G(n, p)$ . In this paper, we consider first order properties and monadic second order properties of bounded *quantifier depth*  $k$ , that is, the length of the longest chain of nested quantifiers in the formula expressing the property. Zero-one laws for properties of quantifier depth  $k$  we call the *zero-one  $k$ -laws*.

The main results of this paper concern the zero-one  $k$ -laws for monadic second order properties (MSO properties). We determine all values  $\alpha > 0$ , for which the zero-one 3-law for MSO properties does not hold. We also show that, contrary to the case of the 3-law, there are infinitely many values of  $\alpha$  for which the zero-one 4-law for MSO properties does not hold. Studying these zero-one laws forces us to analyze the evolution of certain properties of  $G(n, p)$ , which may be of independent interest.

## 1 Introduction

In 1959, P. Erdős and A. Rényi, and independently E. Gilbert, introduced two closely related models for generating random graphs. A seminal paper of Erdős and Rényi [9], that appeared one year later, brought a lot of attention to the subject, giving birth to the vast and ever-developing area of Erdős–Rényi random graphs. In spite of the name, the more popular model  $G(n, p)$  is the one proposed by Gilbert. In this model, we have  $G(n, p) = (V_n, E)$ ,

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where  $V_n = \{1, \dots, n\}$ , and each pair of vertices is connected by an edge with probability  $p$  and independently of other pairs. We are not going to give a survey of this area, but rather refer the reader to the books by B. Bollobás [3] and S. Janson, T. Luczak and A. Rucinski [11], entirely devoted to random graphs, as well as to the book of N. Alon and J. Spencer on probabilistic method [1].

Studying zero-one laws requires an amount, rather modest, of logical prerequisites. We review some of the basics in this paragraph, and refer the reader to [2, 7, 15, 17, 22]. Formulae in the first order language of graphs (FO formulae) are constructed using relational symbols  $\sim$  (the symbol of adjacency) and  $=$ , logical connectives  $\neg, \Rightarrow, \Leftrightarrow, \vee, \wedge$ , variables  $x, y, x_1, \dots$  that express vertices of a graph, and quantifiers  $\forall, \exists$ . Monadic second order, or MSO, formulae (see [10], [16]) are built of the above symbols of the first order language, as well as the variables  $X, Y, X_1, \dots$  that express unary predicates. Following [7], [22], we call the number of nested quantifiers in the longest chain of nested quantifiers of a formula  $\varphi$  the *quantifier depth*  $q(\varphi)$ . We remark that formulae must be finite. For example, the MSO formula

$$(\forall X [(\exists x_1 \exists x_2 [X(x_1) \wedge (\neg X(x_2))]) \Rightarrow (\exists y \exists z [X(y) \wedge (\neg X(z)) \wedge (y \sim z)])])$$

has quantifier depth 3 and expresses the property of being connected. It is known that the property of being connected cannot be expressed by a FO formula (see, e.g., [22]). The quantifier depth of a formula has the following algorithmic interpretation: an FO formula of quantifier depth  $k$  on an  $n$ -vertex graph can be verified in  $O(n^k)$  time.

Many properties of graphs may be expressed via FO formulae. Somewhat surprisingly, Y. Glebskii, D. Kogan, M. Liagonkii and V. Talanov in 1969, and independently R. Fagin in 1976, proved that *any* FO formula is either asymptotically almost surely (a.a.s.) true or a.a.s. false for  $G(n, 1/2)$ , as  $n \rightarrow \infty$ . In such a situation we say that  $G(n, p)$  *obeys the zero-one law* for FO formulae, or *obeys the FO zero-one law*.

More precisely, we say that  $G(n, p)$  *obeys the FO zero-one  $k$ -law* (the *MSO zero-one  $k$ -law*) if any first order formula (monadic second order formula) of quantifier depth  $k$  is either a.a.s. true or a.a.s. false for  $G(n, p)$ . We say that  $G(n, p)$  *obeys the FO zero-one law* (the *MSO zero-one law*) if it obeys the FO zero-one  $k$ -law (the MSO zero-one  $k$ -law) for any positive integer  $k$ .

In 1988 S. Shelah and J. Spencer [12] proved the following zero-one law for the random graph  $G(n, n^{-\alpha})$ .

**Theorem 1.** *Let  $\alpha > 0$ . The random graph  $G(n, n^{-\alpha})$  does not obey the FO zero-one law if and only if either  $\alpha \in (0, 1] \cap \mathbb{Q}$ , or  $\alpha = 1 + 1/\ell$  for some integer  $\ell$ .*

Obviously, there is no MSO zero-one law when even the FO zero-one law does not hold, so we cannot hope for the MSO zero-one law for rational  $\alpha \in (0, 1]$ . In 1993, J. Tyszkiewicz [16]

proved that  $G(n, n^{-\alpha})$  does not obey the MSO zero-one law for irrational  $\alpha \in (0, 1)$  as well. However, if  $\alpha > 1$  and  $\alpha \neq 1 + 1/\ell$  for  $\ell \in \mathbb{N}$ , then the MSO zero-one law holds. This follows via a standard argument in the theory of logical equivalence (see, e.g., [19]). In the next theorem, we summarize the results concerning the MSO zero-one law for  $G(n, n^{-\alpha})$ .

**Theorem 2.** *Let  $\alpha > 0$ . The random graph  $G(n, n^{-\alpha})$  does not obey the MSO zero-one law if and only if either  $\alpha \in (0, 1]$  or  $\alpha = 1 + \frac{1}{\ell}$  for some integer  $\ell$ .*

For a formula  $\varphi$ , we use the notation  $G \models \varphi$  if  $\varphi$  is true for  $G$ . We call the set

$$\mathcal{S}(\varphi) := \{\alpha > 0 : \mathbf{P}(G(n, n^{-\alpha}) \models \varphi) \text{ does not tend to 0 or 1}\}$$

a *spectrum* of  $\varphi$ . J. Spencer proved [14] that there exists an FO formula with infinite spectrum. Moreover, M. Zhukovskii [20] constructed an FO formula with infinite spectrum and of quantifier depth 5.

In this paper, we construct an MSO formula of quantifier depth 4 and with infinite spectrum, and show that this quantifier depth is smallest possible.

**Theorem 3.** *There exists an MSO formula  $\varphi$  with  $q(\varphi) = 4$  and infinite  $\mathcal{S}(\varphi)$ .*

Moreover, we find *all* values of  $\alpha$  for which the MSO zero-one 3-law does not hold.

**Theorem 4.** *The random graph  $G(n, n^{-\alpha})$  does not obey the MSO zero-one 3-law if and only if  $\alpha \in \{\frac{4}{5}, \frac{5}{6}, 1, \frac{9}{8}, \frac{7}{6}, \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, 2\}$ .*

Contrary to the case of the MSO zero-one 3-law, the random graph  $G(n, n^{-\alpha})$  obeys the FO zero-one 3-law for all  $\alpha \in (0, 1)$  (see [21]). If  $\alpha = 1$ , the FO zero-one 3-law fails. By Theorem 1, zero-one 3-law holds for all  $\alpha > 1$  such that  $\alpha \neq 1 + 1/l$  for any positive integer  $l$ . In this paper, we find the remaining part of the full spectrum of FO formulae of quantifier depth 3. It differs slightly from the  $(\alpha > 1)$ -part of the spectrum for MSO formulae of depth 3.

**Theorem 5.** *Let  $\alpha > 1$ . The random graph  $G(n, n^{-\alpha})$  does not obey the FO zero-one 3-law if and only if  $\alpha = 1 + \frac{1}{\ell}$ ,  $\ell \in \{1, 2, 3, 4, 5, 6\}$ .*

Therefore, the minimal  $k$ , for which there exists a FO formula of quantifier depth  $k$  and with infinite spectrum, is either 4 or 5. In [19], it is proven that for any  $k \geq 4$  and any FO formula of quantifier depth  $k$  an intersection of its spectrum with  $(1, \infty)$  is finite. Therefore, it is natural to ask the following question:

**Problem 1.** *Does there exist a FO formula  $\varphi$  of quantifier depth 4 such that  $S(\varphi) \cap (0, 1)$  is infinite?*

Theorems 3, 4, and 5 are proven in Sections 3 and 4. In Section 3, we construct formulae that show that the corresponding zero-one laws in the three theorems do not hold for the declared values of  $\alpha$ . In Section 4, we prove that the zero-one laws from Theorems 4, 5 hold for all the values of  $\alpha$ , not covered in Section 3.

## 2 Preliminaries

We start with the necessary notations and auxiliary statements.

Throughout this paper, we denote the vertex set of a graph  $G$  by  $V(G)$  and its edge set by  $E(G)$  (i.e.,  $G = (V(G), E(G))$ ). For the random graph  $G(n, p)$ , we simply put  $G(n, p) := (V_n, E)$ . The degree of  $v \in V(G)$  is denoted by  $\deg(v)$ .

We need two concentration results about  $G(n, p)$ , embodied in the next two lemmas. The first one concerns the degrees of the vertices in  $G(n, p)$ .

**Lemma 1.** *Let  $p = n^{-\alpha}$  for some  $0 < \alpha < 1$ . Then for some  $c > 0$ , a.a.s. for any  $v \in V_n$  we have*

$$\deg(v) \in \left[ n^{1-\alpha} - cn^{1/2-\alpha/2}\sqrt{\ln n}, n^{1-\alpha} + cn^{1/2-\alpha/2}\sqrt{\ln n} \right].$$

See the proof in [3], Corollary 3.4. The next lemma is concerned with the cardinality  $\alpha(G(n, p))$  of the largest independent set in  $G(n, p)$ .

**Lemma 2.** *Let  $p = n^{-\alpha}$ ,  $0 < \alpha < 1$ . Then a.a.s.*

$$\alpha(G(n, p)) = 2(1 - \alpha)n^\alpha \ln n - 2n^\alpha \ln \ln n + O(n^\alpha).$$

For the proof see [3], Theorem 11.28.

### Small subgraphs of the random graph

Consider a graph  $G$  on  $v$  vertices and  $e$  edges. The fraction  $\rho(G) = \frac{e}{v}$  is called *the density* of  $G$ . Set  $\rho^{\max}(G) = \max_{H \subset G} \rho(H)$ . The threshold probability for the property “ $G(n, p)$  contains  $G$  as a subgraph” was determined by B. Bollobás in 1981 [4, 5]. Moreover, the limiting probability of the property at the threshold was estimated in 1989 [6]. Denote by  $N_G$  the random variable, equal to the number of copies of  $G$  in  $G(n, p)$ .

**Theorem 6.** *If  $p \gg n^{-1/\rho^{\max}(G)}$ , then a.a.s.  $G(n, p)$  contains  $G$ . Moreover,  $\frac{N_G}{\mathbb{E}N_G} \xrightarrow{P} 1$ . If  $p \ll n^{-1/\rho^{\max}(G)}$ , then a.a.s.  $G(n, p)$  does not contain  $G$ . If  $p = cn^{-1/\rho^{\max}(G)}$  for some constant  $c > 0$ , then we have  $N_G > 0$  with asymptotical probability  $q(G)$ , where  $0 < q(G) < 1$ .*

The next lemma follows easily from Theorem 6.

**Lemma 3.** *Let  $\alpha > 0$ . Fix an integer  $k$ , where  $k > 1/\alpha$ , and let  $H$  be a graph on  $k$  vertices. Suppose that  $\rho^{\max}(H) < 1/\alpha$ . Then a.a.s. in  $G(n, n^{-\alpha})$  there exists an induced subgraph  $H'$  isomorphic to  $H$ , such that the vertices of  $H'$  have no common neighbors outside  $H'$ .*

From Theorem 6, it follows that if  $\ell \in \mathbb{N}$  and  $\alpha \in (1 + \frac{1}{\ell+1}, 1 + \frac{1}{\ell})$ , then the following three properties hold:

**T1** The random graph  $G(n, n^{-\alpha})$  is a forest a.a.s.

**T2**( $\ell$ ) A.a.s. any component of  $G(n, n^{-\alpha})$  has at most  $\ell + 1$  vertices.

**T3**( $\ell$ ) For any integer  $K$ , a.a.s. for any tree  $T$  on at most  $\ell + 1$  vertices there are at least  $K$  components in  $G(n, n^{-\alpha})$  which are isomorphic to  $T$ .

If  $\ell \in \mathbb{N}$  and  $\alpha = 1 + \frac{1}{\ell}$ , then the properties T1, T2( $\ell$ ), T3( $\ell - 1$ ) hold, as well as

**T4**( $\ell$ ) For any tree  $T$  on  $\ell + 1$  vertices  $N_T > 0$  with probability  $q(T)$ ,  $0 < q(T) < 1$ .

Finally, we need two extension statements. The first lemma is an easy corollary of Spencer's results [13].

**Lemma 4.** *Let  $\alpha > 0$ . Choose a non-negative integer  $k$ ,  $k < 1/\alpha$ , and a positive integer  $m$ ,  $m \geq k$ . Then a.a.s. for any vertices  $x_1, \dots, x_m$  in  $G(n, n^{-\alpha})$  there is a vertex  $z$  which is adjacent to each of  $x_1, \dots, x_k$  and not adjacent to each of  $x_{k+1}, \dots, x_m$ .*

The second lemma is a particular case of Theorem 2 from [18].

**Lemma 5.** *Let  $\alpha \geq 1/2$ . Choose two integers  $m$  and  $k$ , where  $k \in \{0, 1\}$  and  $m \geq 1$ . Then a.a.s. for any vertices  $x_1, \dots, x_m$  in  $G(n, n^{-\alpha})$  there is a vertex  $z$  which is adjacent to each of  $x_1, \dots, x_k$  and not adjacent to each of  $x_{k+1}, \dots, x_m$ , that additionally satisfies  $(N(z) \cap N(x_i)) \setminus \{x_1, \dots, x_m\} = \emptyset$  for each  $i = 1, \dots, m$ .*

## Ehrenfeucht game

An important tool for many of the results on zero-one laws is the Ehrenfeucht game [1, 11, 15, 22]. The game  $\text{EHR}^{\text{FO}}(A, B, k)$  [2, 7, 8, 17] is played on graphs  $A$  and  $B$ . There

are two players, called Spoiler and Duplicator, and a fixed number of rounds  $k$ . At the  $\nu$ -th round ( $1 \leq \nu \leq k$ ) Spoiler chooses either a vertex  $x_\nu$  of  $A$  or a vertex  $y_\nu$  of  $B$ . Duplicator then chooses a vertex in the other graph.

In the case of monadic second order logic, players can choose subsets as well. Similarly, at the  $\nu$ -th round ( $1 \leq \nu \leq k$ ) of the game  $\text{EHR}^{\text{MSO}}(A, B, k)$  [7, 10, 19] Spoiler chooses one of the  $A, B$ . Say, he chooses  $A$ . Then he either chooses a vertex  $x_\nu$  or a subset  $X_\nu$  of  $V(A)$ . If a vertex is chosen, then Duplicator chooses a vertex of  $B$ . Otherwise, Duplicator chooses a subset of  $V(B)$ .

In  $\text{EHR}^{\text{FO}}$ , at the end of the game the vertices  $x_1, \dots, x_k$  of  $A$ ,  $y_1, \dots, y_k$  of  $B$  are chosen. Duplicator wins if and only if the following property holds.

- For any  $i, j \in \{1, \dots, k\}$ ,  $(x_i \sim x_j) \Leftrightarrow (y_i \sim y_j)$ , and  $(x_i = x_j) \Leftrightarrow (y_i = y_j)$ .

In  $\text{EHR}^{\text{MSO}}$ , at the end of the game vertices  $x_{i_1}, \dots, x_{i_t}$  of  $A$ ,  $y_{i_1}, \dots, y_{i_t}$  of  $B$  and subsets  $X_{j_1}, \dots, X_{j_{k-t}}$  of  $V(A)$ ,  $Y_{j_1}, \dots, Y_{j_{k-t}}$  of  $V(B)$  are chosen. Duplicator wins if and only if the following two properties hold.

- For any  $j_1, j_2 \in \{i_1, \dots, i_t\}$ ,  $(x_{j_1} \sim x_{j_2}) \Leftrightarrow (y_{j_1} \sim y_{j_2})$ , and  $(x_{j_1} = x_{j_2}) \Leftrightarrow (y_{j_1} = y_{j_2})$ .
- For any  $\nu \in \{i_1, \dots, i_t\}$  and  $\mu \in \{j_1, \dots, j_{k-t}\}$ ,  $x_\nu \in X_\mu \Leftrightarrow y_\nu \in Y_\mu$ .

The following well-known result establishes the connection between zero-one laws and Ehrenfeucht games (see, e.g., [10, 11, 15, 22]).

**Theorem 7.** *Let  $k$  be any positive integer. The random graph  $G(n, p)$  obeys the FO zero-one  $k$ -law if and only if a.a.s. Duplicator has a winning strategy in  $\text{EHR}^{\text{FO}}(G(n, p), G(m, p), k)$  as  $n, m \rightarrow \infty$ . The random graph  $G(n, p)$  obeys the MSO zero-one  $k$ -law if and only if a.a.s. Duplicator has a winning strategy in  $\text{EHR}^{\text{MSO}}(G(n, p), G(m, p), k)$  as  $n, m \rightarrow \infty$ .*

### 3 When the zero-one laws fail

This section contains the proofs of the parts of the results, which state that the zero-one law does not hold for some value of  $\alpha$ .

### 3.1 Proof of Theorem 3

A formula  $\varphi$  with  $q(\varphi) = 4$  and infinite spectrum is given below:

$$\begin{aligned} \varphi = \exists X & \\ & ([\exists x_1 \exists x_2 \forall x \quad ([\neg X(x_1)] \wedge X(x_2) \wedge [(X(x) \wedge [x \neq x_2]) \Leftrightarrow ([x \sim x_1] \wedge [x \sim x_2])])]) \wedge \\ & [\neg(\exists z \forall x \quad [X(x) \Rightarrow (\exists v [(v \sim z) \wedge (v \sim x)])])]) \wedge \\ & [\exists x_1 \exists x_2 \exists x_3 \quad (X(x_1) \wedge X(x_2) \wedge X(x_3) \wedge [x_1 \sim x_2] \wedge [x_2 \sim x_3] \wedge [x_1 \sim x_3])]). \end{aligned}$$

Let  $m \in \mathbb{N}$ ,  $\alpha = \frac{1}{2} + \frac{1}{2(m+1)}$  and  $p = n^{-\alpha}$ . Consider the formula

$$\begin{aligned} \varphi^{\text{FO}} = \exists x_1 \exists x_2 ([x_1 \sim x_2] \wedge & \\ [\neg(\exists z \forall x \quad [([x \sim x_1] \wedge [x \sim x_2]) \vee [x = x_1]) \Rightarrow (\exists v [(v \sim z) \wedge (v \sim x)])])]) \wedge & \\ [\exists y_1 \exists y_2 \quad ([x_1 \sim y_1] \wedge [x_1 \sim y_2] \wedge [x_2 \sim y_1] \wedge [x_2 \sim y_2] \wedge [y_1 \sim y_2])]). & \end{aligned}$$

**Lemma 6.** *There exists  $c \in (0, 1)$  such that*

$$\lim_{n \rightarrow \infty} \mathbf{P}(G(n, p) \models \varphi^{\text{FO}}) = c.$$

Proof of Lemma 6 is given in ([20], the proof of Theorem 1). Obviously, the sets of graphs  $\{G \models \varphi\}$ ,  $\{G \models \varphi^{\text{FO}}\}$ , for which formulae  $\varphi$  and  $\varphi^{\text{FO}}$  are true, are the same. Therefore, Lemma 6 holds for  $\varphi$  as well.

Theorem is proven.

### 3.2 The FO zero-one 3-law

Consider  $G(n, p)$  with  $p = n^{-\alpha}$ ,  $\alpha \in \{7/6, 6/5, 5/4, 4/3, 3/2, 2\}$ . Let us prove that  $G(n, p)$  does not obey the FO zero-one 3-law. Below, for each  $\alpha$ , we give a FO formula with quantifier depth at most 3 which is true with asymptotical probability in  $(0, 1)$ . These asymptotical probabilities follow from T1 – T4. Denote

$$P_2(x, y) = ([\exists z (x \sim z) \wedge (y \sim z)] \wedge [x \neq y]), \quad S(x) = [\exists y \exists z ((x \sim y) \wedge (x \sim z) \wedge (y \neq z))].$$

$$1) \quad \alpha = 2: \exists x_1 \exists x_2 (x_1 \sim x_2).$$

$$2) \quad \alpha = \frac{3}{2}: \exists x_1 \exists x_2 P_2(x_1, x_2).$$

$$3) \quad \alpha = \frac{4}{3}: \exists x_1 ([\exists x_2 P_2(x_1, x_2)] \wedge S(x_1)).$$

4)  $\alpha = \frac{5}{4}$ :  $\exists x_1 \varphi(x_1)$ , where

$$\varphi(x_1) = (S(x_1) \wedge [\forall x_2 \exists x_3 ((x_2 \sim x_1) \Rightarrow ((x_2 \sim x_3) \wedge (x_3 \neq x_1)))]).$$

5)  $\alpha = \frac{6}{5}$ :  $\exists x_1 (\varphi(x_1) \wedge [\exists x_2 (P_2(x_1, x_2) \wedge (\exists x_3 ((x_3 \not\sim x_1) \wedge (x_3 \sim x_2)))]))$ .

6)  $\alpha = \frac{7}{6}$ :  $\exists x_1 (\varphi(x_1) \wedge [\forall x_2 (P_2(x_1, x_2) \Rightarrow (\exists x_3 ((x_3 \approx x_1) \wedge (x_3 \sim x_2)))]))$ .

### 3.3 The MSO zero-one 3-law

If  $\alpha = 1$ , then  $G(n, n^{-\alpha})$  does not obey the FO zero-one 3-law, because  $G(n, n^{-\alpha})$  does not contain a triangle with asymptotic probability  $c_1 \in (0, 1)$ .

From Section 3.2, if  $\alpha \in \{7/6, 6/5, 5/4, 4/3, 3/2, 2\}$ , then  $G(n, n^{-\alpha})$  does not obey the FO zero-one 3-law.

Let  $\alpha = 9/8$  and  $p = n^{-\alpha}$ . Consider the formula

$$\begin{aligned} \varphi = \exists X & \\ & ([\exists x (X(x) \wedge [\forall y ([X(y) \wedge (x \neq y)] \Rightarrow [x \sim y])])] \quad \wedge \\ & [\forall x (X(x) \Rightarrow [\exists y ([\neg X(y)] \wedge [x \sim y])])] \quad \wedge \\ & [\forall y ([(\neg X(y)) \wedge (\exists x [X(x) \wedge (x \sim y)])] \Rightarrow [\exists z ([\neg X(z)] \wedge [y \sim z])])]) \quad ). \end{aligned}$$

Obviously, if a graph  $G$  is a forest and  $\varphi$  is true for  $G$ , then there is the following component  $T$  in  $G$ :

$$\begin{aligned} T = (V(T), E(T)), \quad V(T) &= \{t_1^1, t_2^1, t_3^1, t_1^2, t_2^2, t_3^2, t_1^3, t_2^3, t_3^3\}, \\ E(T) &= \{\{t_1^1, t_2^1\}, \{t_2^1, t_3^1\}, \{t_1^1, t_1^2\}, \{t_2^1, t_2^2\}, \{t_3^1, t_3^2\}, \{t_1^2, t_1^3\}, \{t_2^2, t_2^3\}, \{t_3^2, t_3^3\}\}. \end{aligned}$$

Moreover, if there is a component isomorphic to  $T$  in  $G$ , then  $G \models \varphi$ . As  $G(n, p)$  has the properties T1, T2(8), T3(7) and T4(8) (see Section 2), a.a.s.  $G(n, p) \models \varphi$  if and only if in  $G(n, p)$  there is a component isomorphic to  $T$ . By the property T4(8),  $\mathbb{P}(G(n, p) \models \varphi) \rightarrow c_2 \in (0, 1)$  as  $n \rightarrow \infty$ . Therefore,  $G(n, p)$  does not obey the MSO zero-one 3-law.

Let  $\alpha = 5/6$  and  $p = n^{-\alpha}$ . Consider the formula

$$\begin{aligned} \varphi = \exists X & \\ & ([\forall x (X(x) \Rightarrow [(\exists y [X(y) \wedge (x \sim y)]) \wedge (\exists y [X(y) \wedge (x \neq y) \wedge (x \approx y)])])] \quad \wedge \\ & [\exists z ([\neg X(z)] \wedge [\forall x (X(x) \Rightarrow [z \sim x])])]) \quad ). \end{aligned}$$



Let  $H$  be a graph on 5 vertices and 6 edges which is the union of two triangles sharing one vertex. Obviously,  $G \supset H$  if and only if  $G \models \varphi$ . By Theorem 6,  $G(n, p) \supset H$  with asymptotical probability  $c_3 \in (0, 1)$ . Therefore,  $\mathbf{P}(G(n, p) \models \varphi) \rightarrow c_3$  as  $n \rightarrow \infty$  and so  $G(n, p)$  does not obey the MSO zero-one 3-law.

Finally, let  $\alpha = 4/5$  and  $p = n^{-\alpha}$ . Consider the formula

$$\begin{aligned} \varphi = \exists X & \\ & ([\exists x (X(x) \wedge [\forall y ([X(y) \wedge (x \neq y)] \Rightarrow [x \sim y])])]) \quad \wedge \\ & [\exists x \exists y (X(x) \wedge X(y) \wedge [x \neq y] \wedge [x \nsim y])] \quad \wedge \\ & [\exists z ([\neg X(z)] \wedge [\forall x (X(x) \Rightarrow [z \sim x])])] \quad ). \end{aligned}$$

Let  $H$  be a *diamond graph* (two triangles sharing an edge). Obviously,  $G \models \varphi$  if and only if  $G \supset H$ . By Theorem 6,  $\mathbf{P}(G(n, p) \models \varphi) \rightarrow c_4 \in (0, 1)$  as  $n \rightarrow \infty$  and so  $G(n, p)$  does not obey the MSO zero-one 3-law.

## 4 When the zero-one laws hold

Let  $G$  be a graph, and  $X$  be a subset of  $V(G)$ . We denote by  $\bar{X}$  and  $\bar{G}$  the set  $V(G) \setminus X$  and the complement of  $G$  respectively. For a given vertex  $v \in V(G)$ , its set of neighbors is denoted by  $N(v)$ . We denote by  $\deg_X(v)$  the number of vertices from  $X$  that are adjacent to  $v$ . Remark that  $v$  is not adjacent to itself. If  $\deg(v) = 0$ , then we call  $v$  *isolated*.

### 4.1 Classifications and Duplicator's strategy

In this section we describe the Duplicator's winning strategy for the remaining values of  $\alpha$  from Theorems 4 and 5. The strategy in Theorem 4 is based on some classifications of pairs of subsets and pairs subset-vertex, which are obtained in later sections.

In what follows we assume that if Spoiler chooses a set  $X$ , then it is *nontrivial*, that is, both  $|X| \geq 2$  and  $|\bar{X}| \geq 2$ . We treat the special cases  $|X| \in \{0, 1, n-1, n\}$  in what follows. Interchanging  $X, Y$  and  $\bar{X}, \bar{Y}$ , we may w.l.o.g. assume that  $|X| \in \{0, 1\}$ . If  $X = \emptyset$ , then Duplicator chooses  $Y = \emptyset$ , and in the following round plays as if the previous round was not played. If  $X = \{v\}$ , then Duplicator chooses  $Y = \{w\}$ , where  $w$  is taken according to the strategy of Duplicator in case Spoiler chose a vertex  $v$  (and not a set  $\{v\}$ ). In the next round Duplicator plays as if in the previous round the players chose vertices  $v, w$ . Clearly, if Duplicator has a winning strategy for the case when Spoiler chooses either nontrivial sets or vertices, then Duplicator has a winning strategy for the above cases as well.

Note also that in case  $\alpha > 1$ , in view of Theorem 2, we may assume that  $\alpha = 1 + \frac{1}{\ell}$  for  $\ell \in \mathbb{N}$ , which we do tacitly for the rest of the section.

#### 4.1.1 Proof of Theorem 5

For this section we assume that  $\alpha = 1 + \frac{1}{\ell}$ , where  $\ell \geq 7$ . Let  $T$  be a component in  $G$  containing  $x_1$ , the choice of Spoiler in the first round. If in  $H$  there is a component  $F$  isomorphic to  $T$ , then Duplicator chooses  $y_1 = \varphi(x_1)$ , where  $\varphi : T \rightarrow F$  is an isomorphism. In what follows, we discuss the choice of Duplicator in case there is no such component. Note that by T3(6) a.a.s. for any tree on at most 7 vertices there is a component in  $H$  isomorphic to it.

The *distance* between two vertices  $u$  and  $v$  in a connected graph is the minimum length of a path connecting  $u$  and  $v$ . It is denoted by  $d(u, v)$ . If there is no path between  $u$  and  $v$ , we write  $d(u, v) = \infty$  (for any real  $d$ , we let  $d < \infty$ ). For a tree  $R$  and a vertex  $v \in R$  define the *leaf distance set*  $L(v) \subset \mathbb{N} \cup \{0\}$  as the set of all distances from  $v$  to the leaves of  $R$ . Note that  $0 \in L(v)$  iff  $v$  is a leaf itself. We call a subset of  $\mathbb{N} \cup \{0\}$  *admissible*, if it is a leaf distance set for some tree and its vertex. It is easy to see that a set is not admissible iff it contains both 0 and 1, and has cardinality at least 3.

It is not difficult to check that for each admissible  $L$ , there is a tree  $R(L)$  on at most 7 vertices, and a vertex  $v \in R(L)$  in that tree, such that, first,  $L \cap \{0, 1, 2\} = L(v) \cap \{0, 1, 2\}$ , and, second,  $3 \in L(v)$  iff  $L \setminus \{0, 1, 2\}$  is nonempty. Duplicator finds a (tree) component  $K$  in  $H$  isomorphic to  $R(L(x_1))$  and the vertex  $y_1 \in K$  that corresponds to  $x_1$ .

If in the second round Spoiler chooses a vertex (say,  $x_2 \in V(G)$ ), then Duplicator chooses a vertex  $y_2 \in V(H)$  such that

- $(d(y_1, y_2) = d(x_1, x_2)) \Leftrightarrow (d(x_1, x_2) \leq 3)$ ;
- $(d(y_1, y_2) = \infty) \Leftrightarrow (d(x_1, x_2) > 3)$ ;
- if  $d(x_1, x_2) \leq 2$ , then  $y_2$  is a leaf iff  $x_2$  is a leaf;
- $y_2$  is isolated iff  $x_2$  is isolated.

By the choice of  $K$  and  $y_1$ , it is clear that such  $y_2$  exists. It is not difficult to say that in the third round Duplicator has a winning strategy as well. The proof of Theorem 5 is complete.

#### 4.1.2 Pairs of complementary subsets of $V_n$

In the next paragraph, we classify the vertices of  $V_n$  into six types with respect to a subset  $X$  and give the names to the classes.

We say that  $v \in X$  is *X-inside-dominating* (*X-outside-dominating*) if  $v$  is adjacent to all the vertices in  $X$  ( $\bar{X}$ ). We say that  $v \in X$  is *X-inside-isolated* (*X-outside-isolated*) if  $v$  is nonadjacent to all the vertices in  $X$  ( $\bar{X}$ ). Otherwise we say that  $v \in X$  is *X-inside-common* (*X-outside-common*).

In some cases it is more convenient for us not to precise whether  $v \in X$  or  $v \in \bar{X}$ . In this case the classification gets simpler. If  $v \in V$  is adjacent to all vertices of  $X$ , except maybe itself, then we say that  $v$  is *X-dominating*. If  $v \in V$  is nonadjacent to all vertices from  $X$ , then we say that  $v$  is *X-isolated*. Otherwise, we say that  $v$  is *X-common*.

Based on this classification of vertices, we define the *type* of a vertex  $v \in X$ . Two vertices  $v, w \in X$  have the same type if both are *X-dominating* or *X-isolated* or *X-common*, and, similarly, both are  $\bar{X}$ -dominating or  $\bar{X}$ -isolated or  $\bar{X}$ -common. In other words, *the inside type and the outside type of  $v$  and  $w$  are the same*. Note that type is defined with respect to a subset, but it is always clear from the context, w.r.t. which subset the type is defined.

The crucial part of the proof of Theorem 4 is the classification of pairs  $X, \bar{X}$  based on the types their vertices have. The *type of a pair  $X, \bar{X}$*  is specified by the following two parameters:

1. The list of *all* the types that the vertices of  $X$  have.
2. The list of *all* the types that the vertices of  $\bar{X}$  have.

We illustrate the importance of this classification in the next section.

#### 4.1.3 Duplicator's strategy if Spoiler chooses a set in the first round

In this section and Section 4.1.4 we finish the proof of Theorem 4 modulo some classification results, proved in Section 4.2 and 4.3. Let  $G, H$  be two graphs.

**Lemma 7.** *Suppose that in the first round of  $\text{EHR}^{\text{MSO}}(G, H, 3)$  two (nontrivial) subsets  $X \subset V(G)$ ,  $Y \subset V(H)$  are chosen. If the types of the pairs  $X, \bar{X}$  and  $Y, \bar{Y}$  are the same, then in the last two rounds Duplicator has a winning strategy.*

Indeed, if in the second round Spoiler chooses another set, say,  $X' \in V(G)$ , then Duplicator chooses a subset  $Y' \in V(H)$  such that  $Y \cap Y'$  ( $Y \cap \bar{Y}'$ ,  $\bar{Y} \cap Y'$ ,  $\bar{Y} \cap \bar{Y}'$ ) is nonempty iff  $X \cap X'$  ( $X \cap \bar{X}'$ ,  $\bar{X} \cap X'$ ,  $\bar{X} \cap \bar{X}'$ ) is nonempty. Then Duplicator obviously has a winning strategy in the third round.

If in the second round Spoiler chooses a vertex  $v$ , say, in  $X$ , then Duplicator chooses a vertex  $w \in Y$  of the same type as  $v$ . Again, in the third round Duplicator obviously has a winning strategy.

Therefore, to be able to apply Lemma 7, we have to prove that for each admissible  $\alpha$  each type of a pair appears with probability either 0 or 1 in  $G(n, n^{-\alpha})$ . For  $\alpha < 1$  this is done in Section 4.2. When  $\alpha > 1$ , the Duplicator's strategy for choosing the pair of type chosen by Spoiler is described in Section 4.3.

#### 4.1.4 Duplicator's strategy if Spoiler chooses a vertex in the first round

Let  $X \subset V(G)$ ,  $x \in V(G)$ ,  $Y \subset V(H)$ ,  $y \in V(H)$ . We say that the pair  $(x, X)$  is equivalent to the pair  $(y, Y)$ , if

- $x \in X$  iff  $y \in Y$ ;
- the type of  $x$  w.r.t.  $X$  is the same as the type of  $y$  w.r.t.  $Y$  (see Section 4.1.2).

The importance of this definition is justified by the following easy-to-see lemma.

**Lemma 8.** *Suppose that in the first two rounds of  $\text{EHR}^{\text{MSO}}(G, H, 3)$  a vertex  $x$  and a set of vertices  $X$  are chosen in  $G$ , a vertex  $y$  and a set of vertices  $Y$  are chosen in  $H$ . If  $(x, X)$  is equivalent to  $(y, Y)$ , then in the last round Duplicator has a winning strategy.*

We leave the proof of the lemma to the reader. Assume that Spoiler chooses a subset  $X \subset V(G)$  in the second round, and the vertices  $x \in V(G), y \in V(H)$  were chosen in the first round. As in the previous section, it is enough to show that for all admissible values of  $\alpha$  in Theorems 4 and 5 it is a.a.s. possible for Duplicator to choose the subset  $Y \subset V(H)$ , such that the pair  $(y, Y)$  is equivalent to  $(x, X)$  (assuming that Duplicator played "wisely" in the first round as well).

For each  $\alpha \neq 1 + \frac{1}{\ell}$ , where  $\ell = 1, \dots, 6$ , the graph  $G(n, p)$  obeys the FO zero-one 3-law (Theorem 5 and [21]). Therefore, by Theorem 7, a.a.s. Duplicator has a winning strategy in  $\text{EHR}^{\text{FO}}(G(n, p(n)), G(m, p(n)), 3)$  as  $n, m \rightarrow \infty$ . Let  $y_1$  be a vertex chosen by Duplicator according to its (a.a.s.) winning strategy in the first round. If in the second round Spoiler chooses a vertex (say,  $x_2 \in V_n$ ), then Duplicator chooses a vertex  $y_2$  according to his (a.a.s.) winning strategy. Obviously, a.a.s. in the third round Duplicator wins. Assume that in the second round Spoiler chose a set (say,  $X_2 \subset V_n$ ). By Lemma 1, a.a.s.  $\deg(x_1), \deg(y_1) \in [0, n - 4]$ . Moreover, by the strategy described in Section  $\deg(x_1) \geq i$  iff  $\deg(y_1) \geq i$  for  $i = 1, 2$ . Therefore, a.a.s. there exists a set  $Y_2 \subset V_m$  such that  $(x_1, X_2)$  is equivalent to  $(y_1, Y_2)$ . Then Duplicator chooses  $Y_2$ . By Lemma 8, a.a.s. Duplicator has a winning strategy in the last round.

Concluding Section 4.1, we remark that for the proof of Theorem 4 it is sufficient to show that for each  $\alpha > 0$ , where  $\alpha \neq 1 + \frac{1}{\ell}$  for  $\ell \in \{1, \dots, 6\} \cup \{8\}$ , each type of a pair

$X, \bar{X}$  appears with asymptotic probability either 0 or 1 in  $G(n, n^{-\alpha})$ . Then the application of Lemma 7 will finish the proof.

## 4.2 Pairs of subsets for $\alpha < 1$

A subset  $X \subset V$  is *dense* (*sparse*) if it has an  $X$ -inside-dominating vertex ( $X$ -inside-isolated vertex). We say that  $X$  is *complete*, if  $G|_X$  is a complete graph, and *independent*, if  $G|_X$  is an empty graph. Finally, we say that  $X$  is *common* if it does not have neither an inside-dominating nor an inside-isolated vertex.

Note that the set of *inside* types of vertices in  $X$  is determined for  $X$  in each of the five classes above. Therefore, to determine the type of a pair  $X, \bar{X}$ , it is sufficient to determine to which of the two classes each of  $X$  and  $\bar{X}$  belong, and the set of *outside* types of vertices from  $X$  and from  $\bar{X}$ .

In this section, we determine *all* the types of pairs that appear a.a.s. in  $G(n, n^{-\alpha})$  for  $\alpha < 1$ ,  $\alpha \neq 4/5, 5/6$  (which we assume for the rest of Section 4.2), together with the ranges of  $\alpha$  for which they do appear. We do prove that the classification is *complete*, i.e., that all the other types do not appear a.a.s. As we have discussed in Section 4.1.3, this classification is essential for the strategy of Duplicator in the Ehrenfeucht game.

### 4.2.1 Auxiliary lemmas

Let  $p = n^{-\alpha}$ . The next several lemmas will help us to limit the number of possible types.

**Lemma 9.** *Take a subset  $X \subset V_n$  in  $G(n, p)$ . The following bounds on the size of  $X$  hold a.a.s.:*

1. *If  $X$  is complete, then  $|X| \leq \lfloor 2/\alpha \rfloor + 1$ .*
2. *If  $X$  is dense, then  $|X| = O(n^{1-\alpha})$ .*
3. *If  $X$  is sparse, then  $|X| = n - \Omega(n^{1-\alpha})$ .*
4. *If  $X$  is independent, then  $|X| = O(n^\alpha \log n)$ .*

*Proof.* Statement 1 follows from Theorem 6. Both statements 2 and 3 follow from Lemma 1. Statement 4 follows from Lemma 2.  $\square$

Lemma 9 allows us to exclude some pairs of types of sets that cannot form a pair  $X, \bar{X}$ . E.g., two dense sets cannot form a pair  $X, \bar{X}$  since the sum of their cardinalities is much smaller than  $n$ . The pair “dense–sparse” is not excluded by Lemma 9, but is impossible in most situations, as shown in the next lemma.

- Lemma 10.** 1. A.a.s. there are no two distinct vertices  $u, v$  in  $G(n, p)$  with  $N(u) \subset N(v)$ .  
2. A.a.s. in  $G(n, p)$  there are no two distinct vertices  $u, v$  such that  $v$  is an  $X$ -dominating vertex and  $u$  is an  $\bar{X}$ -dominating or  $\bar{X}$ -isolated vertex for some  $X \subset V_n$ .  
3. If  $\alpha < 1/2$ , then a.a.s. in  $G(n, p)$  there are no two distinct vertices  $u, v$ , such that  $v$  is  $X$ -isolated and  $u$  is  $\bar{X}$ -isolated for some  $X \subset V_n$ .

*Proof.* 1. The probability that for some  $u, v \in V$  we have  $N(u) \subset N(v)$  is at most

$$n^2(1 - p(1 - p))^{n-2} \leq n^2 e^{-(n-2)p(1-p)} = o(1) \text{ as } n \rightarrow \infty.$$

2. By Lemma 1, a.a.s.  $\deg v + \deg u = O(n^{1-\alpha})$  for all pairs  $u, v \in V_n$ . Therefore, a.a.s. there are no pairs  $v, w \in V$  where  $v$  is  $X$ -dominating and  $u$  is  $\bar{X}$ -dominating. The second possibility from the formulation is ruled out by Statement 1 of this lemma.

3. The probability that there exist  $u, v \in V_n$  such that  $u$  is  $X$ -isolated and  $v$  is  $\bar{X}$ -isolated, is at most  $n^2(1 - p^2)^{n-2} \leq n^2 e^{-p^2(n-2)}$ . This expression tends to zero if  $\alpha < 1/2$  and  $n \rightarrow \infty$ .  $\square$

If  $X$  is dense, then  $X$  has a dominating vertex, so  $\bar{X}$  cannot have isolated vertices, and therefore cannot be sparse, unless there is a vertex  $v$  which is both dominating for  $X$  and isolated for  $\bar{X}$ . If  $v$  is dominating for  $X$  and isolated for  $\bar{X}$ , or vice versa, then we say that  $v$  is  $X$ -special. Note that due to the concentrations of degrees (Lemma 1) a.a.s. no vertex is both dominating for both  $X$  and  $\bar{X}$ , and no vertex is both isolated for both  $X$  and  $\bar{X}$ .

#### 4.2.2 Classification of subsets without special vertices

For a moment we consider only the sets  $X$  for which there are no special vertices  $v$ . The somewhat special case of  $X$ -special vertices we treat later.

The graph in Figure 1 represents the possibilities that are not ruled out by Lemma 9 and statement 2 of Lemma 10. The vertices are the five possible types of subsets, and the edges are represented by dashed lines. A non-edge between two vertices means that there cannot be a pair  $X, \bar{X}$  where  $X$  and  $\bar{X}$  belong to the types of sets represented by the corresponding vertices (in the assumption that there are no  $X$ -special vertices). Note that there are two loops in the graph.

The list below the name of the type give the possible types of vertices from the set. All the types of pairs of sets and/or vertices from the set that do not appear on the scheme do not appear in  $G(n, p)$  a.a.s. (assuming that the set does not have special vertices). Complete and independent sets can have only outside-common vertices, since any non-outside-common vertex in a complete or independent set is special.

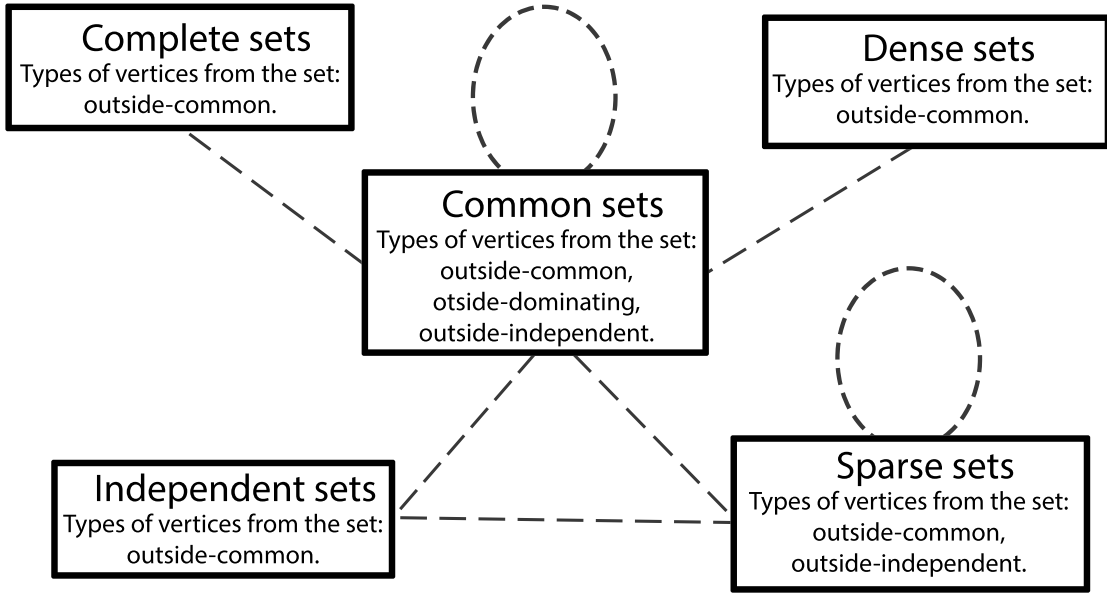


Figure 1. The pairs of  $X, \bar{X}$  and the types of their vertices that are allowed by Lemmas 9, 10.

Unfortunately, this graph does not convey all the information that we need: it does not specify, which types of vertices can appear *simultaneously* in  $X$  or in  $\bar{X}$ , provided that  $X, \bar{X}$  are of a given type.

**Lemma 11.** *A.a.s. any nontrivial subset  $X \subset V_n$  has an  $X$ -outside-common vertex.*

*Proof.* Let  $X \subset V$  be a nontrivial set of cardinality  $i$ . The probability that one of the vertices of  $X$  is outside-common is  $1 - (p^{n-i} + (1-p)^{n-i})^i$ . Therefore, the probability that there exists a nontrivial  $X$  with no  $X$ -common vertices is at most

$$\sum_{i=2}^{n-2} \binom{n}{i} (p^{n-i} + (1-p)^{n-i})^i \leq \sum_{i=2}^{n-2} \binom{n}{i} ((1-p)^{n-i-1})^i \leq 2 \sum_{i=2}^{\lfloor n/2 \rfloor} \binom{n}{i} ((1-p)^{n-i-1})^i \leq$$

$$2 \sum_{i=2}^{\lfloor n/2 \rfloor} (ne^{-p(\lfloor n/2 \rfloor + 1)})^i < \frac{2ne^{-p(\lfloor n/2 \rfloor + 1)}}{(1-p)(1 - ne^{-p(\lfloor n/2 \rfloor + 1)})} = o(1) \quad \text{for } n \rightarrow \infty.$$

□

**Lemma 12.** *A.a.s. for any complete  $X$  there is an  $\bar{X}$ -outside-sparse vertex, and there is  $K_3$  in  $G(n, p)$ .*

*Proof.* By Theorem 6, a.a.s. any complete subgraph of  $G(n, p)$  has at most  $\lceil 2/\alpha \rceil + 1$  vertices. By Lemma 4, a.a.s., for any  $X$  such that  $G(n, p)|_X$  is complete, there is an  $\bar{X}$ -outside-sparse vertex. Finally, by Theorem 6, a.a.s. there is a copy of  $K_3$  in  $G(n, p)$ .  $\square$

In Table 1, we give all possible types of pairs of  $X$  and  $\bar{X}$  in  $G(n, n^{-\alpha})$  that do not have a special vertex, up to a permutation of  $X$  and  $\bar{X}$ .

Each type in the table is assigned a range of values of  $\alpha$ . Any type appears a.a.s. in  $G(n, n^{-\alpha})$  iff it is present in the table and the value of  $\alpha$  belongs to the prescribed range. Otherwise, the type a.a.s. do not appear in  $G(n, n^{-\alpha})$ .

The table reflects the restrictions that are imposed on the types of pairs by Figure 1 and Lemmas 10, 11 and 12. It is not difficult, although a bit tedious, to verify that the 26 cases of the table are *exactly* the ones that are left after applying the aforementioned statements.

The ranges of  $\alpha$  are, however, unexplained in many cases. Therefore, we are left to show that, first, outside the specified ranges of  $\alpha$  the corresponding types a.a.s. do not appear in  $G(n, n^{-\alpha})$ , and that, second, inside the ranges they a.a.s. do appear. But first we discuss the case of special vertices.

### 4.2.3 Special vertices

Assume that  $X$  has a special vertex  $v$ . If  $v$  is special for  $X$ , it is special for  $\bar{X}$  as well. Therefore, we may assume that  $v \in X$ .

We have two cases to consider: the case when  $v$  is  $X$ -isolated and  $\bar{X}$ -dominating, and the case when  $X$  and  $\bar{X}$  are interchanged. By Lemma 10, there are no other  $X$ -dominating or  $\bar{X}$ -dominating vertices, as well as other special vertices for  $X$ .

**Case 1** If  $v$  is  $X$ -isolated and  $\bar{X}$ -dominating, then, by Statement 1 of Lemma 10, there are no other  $X$ -isolated vertices. In particular, all vertices of  $\bar{X}$  are  $\bar{X}$ -outside-common.

**Case 2** If  $v$  is  $X$ -dominating and  $\bar{X}$ -isolated, then there are no other  $\bar{X}$ -isolated vertices, as well as  $X$ -inside-isolated vertices.

We summarize the remaining cases in a smaller table, resembling Table 1. Remark that the type of  $X$  is determined by the type of the special vertex, so it is not listed in Table 2. Moreover, we list outside types of not special vertices only. We remark that in this case the limitations  $\alpha \in [1/2, 1)$  are explained by the application of Lemma 10: for  $\alpha < 1/2$  there are no two vertices  $u, v$ , such that  $u$  is  $X$ -isolated and  $v$  is  $\bar{X}$ -isolated.

As in the case of Table 1, we need to prove that the types listed in Table 2 are a.a.s. present in  $G(n, p)$  within the ranges present, and that the other types a.a.s. do not appear. Note that some of the cases have  $\emptyset$  as the range of  $\alpha$ , which means that they a.a.s. do not appear for any  $\alpha$ . We have left them in the table since they are not ruled out by the previous considerations.



	Type of subsets $X \mid \bar{X}$	Outside types of vertices in $X \mid \bar{X}$	$\alpha$
1	complete   common	common   dominating, isolated, common	$(0, 1)$
2	complete   common	common   isolated, common	$(0, 1)$
3	dense   common	common   dominating, isolated, common	$(0, 4/5)$
4	dense   common	common   isolated, common	$(0, 1)$
5	dense   common	common   dominating, common	$(0, 1/3)$
6	dense   common	common   common	$(0, 1/2)$
7	independent   common	common   dominating, isolated, common	$(0, 1)$
8	independent   common	common   isolated, common	$(0, 1)$
9	independent   common	common   dominating, common	$(0, 1/3)$
10	independent   common	common   common	$(0, 1)$
11	independent   sparse	common   isolated, common	$[2/3, 1)$
12	independent   sparse	common   common	$[2/3, 1)$
13	sparse   sparse	isolated, common   isolated, common	$[1/2, 1)$
14	sparse   sparse	isolated, common   common	$[1/2, 1)$
15	sparse   sparse	common   common	$[1/2, 1)$
16	sparse   common	isolated, common   isolated, common	$[1/2, 1)$
17	sparse   common	isolated, common   common	$[1/2, 1)$
18	sparse   common	common   common	$(0, 1)$
19	sparse   common	common   isolated, common	$(0, 1)$
20	sparse   common	common   dominating, common	$(0, 1/2)$
21	sparse   common	common   dominating, isolated, common	$(0, 1)$
22	common   common	isolated, common   isolated, common	$[1/2, 1)$
23	common   common	common   common	$(0, 1)$
24	common   common	common   isolated, common	$(0, 1)$
25	common   common	common   dominating, common	$(0, 1/2)$
26	common   common	common   dominating, isolated, common	$(0, 5/6)$

Table 1: Types of pairs of subsets without special vertices.

Actually, Lemma 5 is sufficient to explain most of Table 2, via the following corollary:

**Corollary 1.** *For  $\alpha \geq 1/2$  a.a.s. for any vertex  $v$  there are two vertices  $u, w$ , such that  $u \sim v, w \not\sim v$  and  $N(v) \cap N(u) = \emptyset, N(v) \cap N(w) = \emptyset$ .*

Choose  $\alpha \in [1/2, 1)$  and take a vertex  $v$  that lies in a triangle guaranteed by Lemma 12.

		Type of $\bar{X}$	Outside types of vertices in $X \mid \bar{X}$	$\alpha$
Case 1	1.1	sparse	common, isolated   common	$[1/2, 1)$
	1.2	sparse	common   common	$\emptyset$
	1.3	independent	common, isolated   common	$[2/3, 1)$
	1.4	independent	common   common	$\emptyset$
	1.5	common	common, isolated   common	$\emptyset$
	1.6	common	common   common	$(0, 1/2)$
Case 2	2.1	common	common   common, isolated	$[1/2, 1)$
	2.2	common	common   common	$(0, 1/2)$

Table 2: Types of pairs of subsets.

Putting  $\bar{X} := N(v)$ , the corollary above guarantees an  $X$ -outside-isolated and an  $\bar{X}$ -inside-isolated vertex. Thus, 1.1 is explained, as well as 1.2: type 1.2 is impossible for  $\alpha < 1/2$  since  $\bar{X}$  cannot be sparse, and it is impossible for  $\alpha \geq 1/2$  since then there must be an  $X$ -outside-isolated vertex. Similarly, types 1.4 and 1.5 are ruled out as well. The explanation in case 2.1 is the same as in case 1.1.

We are left with types 1.3, 1.6 and 2.2. The existence of 1.6 and 2.2 is simple: since for  $\alpha < 1/2$  we cannot have neither  $X$ -isolated nor  $\bar{X}$ -isolated vertices other than the special vertex, taking a vertex  $v$  and putting  $\bar{X} := N(v)$  and  $X := N(v) \cup \{v\}$ , we get types 1.6 and 2.2, respectively. For  $\alpha \geq 1/2$ , the existence of either  $X$ -isolated or  $\bar{X}$ -isolated vertices is guaranteed by Corollary 1, so these types do not appear for  $\alpha \geq 1/2$ .

The a.a.s. existence of type 1.3 within the range  $\alpha \in [2/3, 1)$  (as well as its a.a.s. nonexistence outside this range) is explained by the following lemma, combined with Corollary 1.

**Lemma 13.** *If  $\alpha < 2/3$ , then a.a.s. there is no vertex  $v$  with  $N(v)$  being an independent set. If  $\alpha \geq 2/3$ , then a.a.s. there is such a vertex  $v$ .*

*Proof.* Let  $\alpha < 2/3$ . Fix a vertex  $v \in V$ . Let  $X$  be a set of neighbors of  $v$  in  $G(n, p)$ . The probability  $\gamma = \gamma(X)$  of  $X$  being an independent set equals

$$\sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} \leq \sum_{k=0}^{n-1} \left( \frac{enp(1-p)^{\frac{n}{k}}}{k} \right)^k (1-p)^{\frac{k^2}{2} - \frac{3k}{2} - 1} \leq$$

$$\sum_{k=0}^{n-1} \left( \frac{enp}{k} e^{-\frac{np}{k}} \right)^k (1-p)^{\frac{k^2}{2} - \frac{3k}{2} - 1} \leq \sum_{k=0}^{\lceil p^{-1/2} \log(1/p) \rceil} \left( \frac{enp}{k} e^{-\frac{np}{k}} \right)^k + \sum_{k=\lceil p^{-1/2} \log(1/p) \rceil}^{n-1} (1-p)^{\frac{k^2}{2} - \frac{3k}{2} - 1}.$$

In the last transition we used the fact that  $ex \leq e^x$  for any  $x$ . We also have  $ex \leq e^{(\frac{1}{2}+o(1))x}$ , and thus the first sum in the last displayed expression is bounded by  $ne^{-\Omega(np^{3/2}/\ln(1/p))} \leq ne^{-n^\beta}$  for some  $\beta > 0$ , since  $\alpha < 2/3$ . This expression is obviously  $o(1/n)$ . The second sum in the last displayed equation is  $O(ne^{-p(p^{-1}(\ln(1/p))^2)}) = o(1/n)$ , since  $1/p = n^\alpha$  for some  $\alpha > 0$ . Therefore, a.a.s. there is no vertex  $v$  in  $V_n$  with  $N(v)$  being an independent set.

If  $\alpha > 2/3$ , then

$$\gamma \geq \sum_{k=\lfloor 1/2n^{1-\alpha} \rfloor}^{\lfloor 3/2n^{1-\alpha} \rfloor} \binom{n-1}{k} p^k (1-p)^{n-1-k+\binom{k}{2}} = (1+o(1)) \sum_{k=\lfloor 1/2n^{1-\alpha} \rfloor}^{\lfloor 3/2n^{1-\alpha} \rfloor} \binom{n}{k} p^k (1-p)^{n-k} = 1+o(1).$$

If  $\alpha = 2/3$ , then

$$\gamma \geq (1+o(1)) \sum_{k=\lfloor 1/2n^{1/3} \rfloor}^{\lfloor 3/2n^{1/3} \rfloor} \binom{n}{k} p^k (1-p)^{n-k+k^2/2} = (1-p)^{9/4n^{2/3}} (1+o(1)) \rightarrow e^{-9/4}, \quad n \rightarrow \infty.$$

Lemma 5 implies that for each  $k = 2, 3, \dots$  we have a set of vertices  $v_1, \dots, v_k$  such that  $v_i$ 's form an independent set and, moreover,  $N(v_i) \cap N(v_j) = \emptyset$  for each  $1 \leq i < j \leq k$ . The event “ $N(v_i)$  is an independent set” is independent of events “ $N(v_j)$  is an independent set” for  $j \neq i$ . Therefore, for any  $\varepsilon > 0$  there exists  $k$  such that

$$\mathbb{P}(\exists m \in \{1, \dots, k\} \text{ } X_m \text{ is an independent set}) \geq (1 - \varepsilon) + o(1).$$

This means that a.a.s. in  $G(n, p)$  there exists  $v$  such that  $N(v)$  is an independent set.  $\square$

#### 4.2.4 No special vertices

We return to the case when  $X$  does not have a special vertex. We recall that we have to prove the existence of types 1 – 26 from Table 1 strictly within the mentioned ranges of  $\alpha$ . It is easy to see that in all cases when the range of a type is  $\alpha \in [1/2, 1)$ , the nonexistence of the type outside the range is explained using Lemma 10: for  $\alpha < 1/2$ , a.a.s. there are no two vertices  $u, v$ , such that  $u$  is  $X$ -isolated and  $v$  is  $\bar{X}$ -isolated. Similarly, if the range for a given type is  $(0, 1/2)$ , then the nonexistence of this type for  $\alpha \geq 1/2$  is explained by Corollary 1: in each case we have a  $X$ -dominating vertex, and thus we must have an  $\bar{X}$ -outside-isolated vertex, guaranteed by Corollary 1.

For all types we make use of Lemma 11, that guarantees the presense of  $X$ -outside-common and  $\bar{X}$ -outside-common vertices in all cases. In what follows we do not repeat this

reference.

**Types 1, 2.**

In these cases we have to prove the a.a.s. existence only. Take as  $X$  an edge of a triangle for type 1 and a maximal clique for type 2. By Lemma 12, both types a.a.s. exist in  $G(n, p)$ .

**Types 3 – 6.**

For these cases we need the following auxiliary lemmas.

**Lemma 14.** *A.a.s. in  $G(n, p)$  for any  $2\lfloor n^\alpha \ln \ln n \rfloor$  vertices there is a vertex which is not adjacent to any of them. Fix  $2\lfloor n^\alpha \ln n \rfloor$  vertices. A.a.s. any other vertex is adjacent to at least one of them and not adjacent to at least one of them.*

*Proof.* Set  $m = 2\lfloor n^\alpha \ln \ln n \rfloor$ . Consider vertices  $v_1, \dots, v_m$ . With probability

$$(1 - (1 - p)^m)^{n-m} \leq e^{-(n-m)/\ln^4 n}$$

each vertex  $v \in V_n \setminus \{v_1, \dots, v_m\}$  is adjacent to some of  $v_1, \dots, v_m$ . Therefore, for any  $m$  vertices there is a vertex which is not adjacent to any of them with the probability at least  $1 - n^m e^{-(n-m)/\ln^4 n} = 1 + o(1)$ .

Set  $m = 2\lfloor n^\alpha \ln n \rfloor$  and choose distinct vertices  $v_1, \dots, v_m$ . Any other vertex is adjacent to at least one of  $v_1, \dots, v_m$  and not adjacent to at least one of them with the probability

$$(1 - (1 - p)^m - p^m)^{n-m} \geq e^{-(1+o(1))/n} = 1 + o(1).$$

□

**Lemma 15.** *If  $\alpha \geq 1/3$ , then a.a.s.  $G(n, p)$  does not contain a complete bipartite graph  $K_{2, \lceil n^{1/3} \ln \ln n \rceil}$  as a subgraph.*

*Proof.* Denote  $m = \lceil n^{1/3} \ln \ln n \rceil$ . Such a subgraph exists with a probability at most

$$\binom{n}{m} p^{2m} \leq \frac{n^m e^m}{m^m} p^{2m} = e^{m(\ln n(1-2\alpha)+1-\ln m)} \leq e^{m(1-\ln \ln n)} = o(1).$$

□

**Lemma 16.** *If  $\alpha < 1/3$ , then a.a.s. in  $G(n, p)$  any two vertices have at least  $\lceil n^{1/3} \rceil$  common neighbors.*

*Proof.* For fixed two vertices  $u, v$ , denote by  $X_i(u, v)$  the indicator of the event  $i \sim u, i \sim v$ . From Chernoff bound,

$$\mathbb{P} \left( \exists u, v \left( \sum_{i=1}^n X_i(u, v) < n^{1/3} \right) \right) \leq n^2 e^{-\frac{((n-2)p^2 - n^{1/3})^2}{2(n-2)p^2}} = o(1).$$

□

We go on to the analysis of types 3 – 6. If  $\alpha < 4/5$ , then by Theorem 6 a.a.s.  $G(n, p)$  contains an induced subgraph on 4 vertices and 5 edges: the vertices  $v_1, \dots, v_4$ , and all pairs of vertices are adjacent except  $v_1, v_2$ . Put  $X := \{v_1, v_2, v_3\}$ . It has a dominating vertex  $v_4$ . By Lemma 4, a.a.s.  $X$  has type 3.

Any dense, but not complete, set  $X$  has cardinality at least 3. Moreover, if it has an  $\bar{X}$ -outside-dominating vertex  $v$ , then the induced graph on  $X \cup \{v\}$  has density at least  $5/4$ . Therefore, if  $\alpha > 4/5$ , then by Theorem 6 a.a.s. in  $V_n$  there is no such  $X$ , and, consequently, no pair of type 3.

Let  $s = \lfloor 1/\alpha \rfloor$ . For any  $\alpha \in (0, 1)$ , by Lemma 3 a.a.s. in  $G(n, p)$  there is  $X \subset V_n$  with  $|X| = s + 1$ , such that  $G(n, p)|_X$  is a star with no  $\bar{X}$ -outside-dense vertices. Lemma 4 guarantees that it has an  $\bar{X}$ -outside-sparse vertex, and so  $X$  is of type 4.

If  $\alpha < 1/2$ , then by Lemma 1 a.a.s. for  $v \in V_n$  we have  $|N(v)| > 2\sqrt{n}$ . Choose  $X = \{v\} \cup X_0$ , where  $X_0 \subset N(v)$ ,  $|X| = \lfloor \sqrt{n} \rfloor$ . By Lemma 14, a.a.s. any vertex of  $\bar{X}$  is  $\bar{X}$ -outside-common, so  $X$  is a.a.s. of type 6.

By Lemmas 14, 15, if  $\alpha \geq 1/3$ , then in  $G(n, p)$  there are no subsets of type 5. Let  $\alpha < 1/3$ . Fix an ordering on the set of pairs of vertices of  $V_n$  and take the first edge  $uv$  of  $G(n, p)$  in this ordering. By Lemma 16, a.a.s.  $u$  and  $v$  share at least  $n^{1/3}$  neighbors. Denote  $X := \{u\} \cup (N(u) \cap N(v))$ . By Lemma 14, a.a.s., all vertices in  $\bar{X} \setminus \{v\}$  are  $\bar{X}$ -outside-common. Moreover, by Lemma 10 neither  $v$  nor  $u$  is  $X$ -special. Therefore, a.a.s.  $X$  has type 5.

## Types 7 – 12.

This is the most difficult situation. For these cases we need the following auxiliary lemmas.

**Lemma 17.** *Consider the event  $A$  “there exists an independent set  $X$  such that there is an  $\bar{X}$ -outside-dominating vertex but there are no  $\bar{X}$ -outside-isolated vertices”. If  $\alpha \geq 1/3$ , then a.a.s.  $A$  does not hold. If  $\alpha < 1/3$ , then a.a.s.  $A$  holds.*

*Proof.* Let  $A_k$  be the number of independent sets  $X$  of size  $k$  together with a vertex  $v$ , such that  $v$  is  $X$ -dominating but there are no  $\bar{X}$ -outside-isolated vertices. By Lemma 14, it is sufficient to restrict our attention to the case  $k \in I := [2n^\alpha \ln \ln n, 2n^\alpha \ln n]$ . For each  $k$  we have

$$\mathbb{E}A_k = \binom{n}{k} (n-k)p^k (1-p)^{\binom{k}{2}} (1 - (1-p)^k)^{n-k-1}. \quad (1)$$

Let us first prove that  $\sum_{k \in I} \mathbb{E}A_k \rightarrow 0$  if  $\alpha \geq \frac{1}{3}$ . This will obviously imply the first part of the lemma. Choose  $\alpha \geq \frac{1}{3}$ . Assuming that  $k \in I$  and putting  $k = n^\alpha x \ln n$ , we get

$$\mathbb{E}A_k \leq \left(\frac{ne}{k}\right)^k np^k e^{-p\binom{k}{2}} (1 - e^{-pk - p^2k})^{n-k-1} = e^{k(\ln n - \ln k - \ln p - \frac{1}{2}pk) - (1+o(1))ne^{-pk} + O(k)} \leq$$

$$e^{k(1-2\alpha-\frac{x}{2})\log n - k\ln\ln n - (1+o(1))n^{1-x} + O(k)} := f(k).$$

If for some constant  $c$  we have  $1-x \geq c > \alpha$ , then, obviously,  $f(k) = o(1/n)$ . Therefore, we may assume that  $x \geq 1-\alpha+o(1)$ . In that case we have  $1-2\alpha-\frac{x}{2} \leq \frac{1}{2}-\frac{3}{2}\alpha+o(1)$ , and thus  $f(k) = o(1/n)$  if  $\alpha > \frac{1}{3}$ . If  $\alpha = \frac{1}{3}$ , then  $1-2\alpha-\frac{x}{2} = \frac{1}{3}-\frac{x}{2}$ , so  $f(k) = o(1/n)$  if  $x \geq \frac{2}{3}$ . Therefore, we assume that  $\alpha = \frac{1}{3}$  and  $x = \frac{2}{3} + o(1)$ .

If  $x \leq \frac{2}{3} - 2\frac{\ln\ln n}{\ln n}$ , then

$$k\left(\frac{1}{3} - \frac{x}{2}\right)\ln n - (1+o(1))n^{1-x} = n^{1/3}x\left(\frac{1}{3} - \frac{x}{2}\right)(\ln n)^2 - (1+o(1))n^{1/3}(\ln n)^2 < 0,$$

so in this case  $f(k) \leq e^{-k\ln\ln\ln n + O(k)} = o(1/n)$ .

If  $\frac{2}{3} - \frac{3}{2}\frac{\ln\ln n}{\ln n} \geq x > \frac{2}{3} - 2\frac{\ln\ln n}{\ln n}$ , then  $k(\frac{1}{3} - \frac{x}{2})\ln n < k\ln\ln n$ , so  $f(k) \leq e^{-(1+o(1))n^{1-x} + O(k)} = o(1/n)$ , since  $k = o(n^{1-x})$ .

If  $x > \frac{2}{3} - \frac{3}{2}\frac{\ln\ln n}{\ln n}$ , then  $k(\frac{1}{3} - \frac{x}{2})\ln n < \frac{3}{4}k\ln\ln n$ , so  $f(k) \leq e^{-\frac{1}{4}k\log\log n + O(k)} = o(1/n)$ . Summing over  $k \in I$  the bounds that we obtained, we get that  $\sum_{k \in I} \mathbb{E}A_k = o(1)$  for  $\alpha \geq 1/3$ .

Let  $\alpha < 1/3$ . We prove that  $\mathbb{P}(A_k > 0) \rightarrow 1$  as  $n \rightarrow \infty$ , where  $k = (1-\alpha)n^\alpha \ln n$ . This follows from Chebyshev's inequality, since  $\mathbb{E}A_k \rightarrow \infty$  as  $n \rightarrow \infty$  and  $\mathbb{D}A_k = o((\mathbb{E}A_k)^2)$ . The expectation is calculated in (1). Moreover, the upper bound on  $\mathbb{E}A_k$ , proven after (1), is also a lower bound, if one adds a factor  $(1+o(1))$  into the exponent (note that  $x = 1-\alpha$  in our case):

$$\mathbb{E}A_k \geq e^{(1+o(1))\left(k(1-2\alpha-\frac{1-\alpha}{2})\ln n - k\ln\ln n - (1+o(1))n^\alpha + O(k)\right)} = e^{(\frac{1}{2}+o(1))(1-3\alpha)k\ln n}.$$

This expression obviously tends to infinity. Next, we show that  $\mathbb{E}A_k^2 \leq (1+o(1))(\mathbb{E}A_k)^2$ . We have

$$\mathbb{E}A_k^2 \leq \sum_{j=0}^k \binom{n}{k} \binom{n-k}{k-j} \binom{k}{j} (n-k)^2 p^{2k-j} (1-p)^{2\binom{k}{2}-\binom{j}{2}} (1-2(1-p)^k + (1-p)^{2k-j})^{n-2k-2}.$$

Let us denote the  $j$ -th summand of the above expression by  $g(j)$ . It is easy to see that  $g(0) = (1+o(1))(\mathbb{E}A_k)^2$ . Indeed,

$$\frac{g(0)}{(\mathbb{E}A_k)^2} \leq (1 - (1-p)^k)^{-4k-4} \leq \exp(O(k)e^{-pk(1+o(1))}) = \exp(n^{-(1+o(1))(1-2\alpha)}) = 1 + o(1).$$

Thus, it is sufficient for us to show that for each  $j \geq 1$  the expression  $g(j)$  is much smaller than  $g(0)$ .

Let us first consider the case  $j \leq \alpha k$ . Then

$$\frac{(1 - 2(1 - p)^k + (1 - p)^{2k-j})^{n-2k-2}}{(1 - 2(1 - p)^k + (1 - p)^{2k})^{n-2k-2}} \leq (1 + (1 + o(1))(1 - p)^{2k-j})^n \leq (1 + (1 + o(1))e^{-(2-\alpha)kp})^n.$$

We have  $(2 - \alpha)kp = (2 - \alpha)(1 - \alpha) \ln n > c \log n$ , where  $c > 1$ . Consequently, for  $n$  large enough, the last expression in the displayed formula is at most  $(1 + \frac{1}{n^c})^n = 1 + o(1)$ . Therefore, in this case,

$$\frac{g(j)}{g(0)} \sim \frac{\binom{n-k}{k-j} \binom{k}{j}}{\binom{n}{k} p^j (1-p)^{\binom{j}{2}}} = O\left(\frac{k^j \left(\frac{ke}{j}\right)^j}{n^j p^j e^{-(\frac{1}{2}+o(1))pj^2}}\right) \leq \frac{1}{n^{(1-3\alpha+o(1))j} j j e^{-(\frac{1}{2}+o(1))pj^2}}.$$

Since  $j \leq \alpha k = \alpha(1 - \alpha)n^\alpha \ln n$ , we have  $j^j e^{-(\frac{1}{2}+o(1))pj^2} = e^{j(\log j - (\frac{1}{2}+o(1))pj)} = \Omega(1)$ . Therefore, for each  $j \leq \alpha k$ , we have  $\frac{g(j)}{g(0)} \leq n^{-(1-3\alpha+o(1))j}$ .

Next, consider the case  $j \geq \alpha k$ . In this case, we use the following crude estimate:

$$\frac{(1 - 2(1 - p)^k + (1 - p)^{2k-j})^{n-2k-2}}{(1 - 2(1 - p)^k + (1 - p)^{2k})^{n-2k-2}} \leq (1 + (1 + o(1))(1 - p)^k)^n \leq e^{(1+o(1))ne^{-kp}} = e^{(1+o(1))n^\alpha}.$$

Remark that for  $j \geq \alpha k$  we have  $e^{(1+o(1))n^\alpha} = n^{o(j)}$ ,  $j^j = n^{(1+o(1))\alpha j}$ . Finally, we have

$$\frac{g(j)}{g(0)} = O\left(\frac{k^j \left(\frac{ke}{j}\right)^j e^{(1+o(1))n^\alpha}}{n^j p^j e^{-(\frac{1}{2}+o(1))pj^2}}\right) = \frac{1}{n^{(1-2\alpha+o(1))j} e^{-(\frac{1}{2}+o(1))pj^2}}.$$

Since  $j \leq k$ , we have

$$e^{-(\frac{1}{2}+o(1))pj^2} \geq e^{-(\frac{1}{2}+o(1))(1-\alpha)j \ln n} = n^{-(\frac{1}{2}-\frac{\alpha}{2}+o(1))j}.$$

Therefore,  $\frac{g(j)}{g(0)} = n^{-(\frac{1}{2}-\frac{3\alpha}{2}+o(1))j}$  for each  $j \geq \alpha k$ . We conclude that there exists  $\delta > 0$  such that  $\sum_{j=1}^k g(j) \leq \sum_{j=1}^k n^{-\delta j} g(0) = o(g(0))$ , and so  $\mathbb{E}A_k^2 \leq (1 + o(1))(\mathbb{E}A_k)^2$ .  $\square$

**Lemma 18.** *A.a.s. in  $G(n, p)$  there exists an independent set  $X$  such that  $\bar{X}$  is a common set, and all its vertices are  $\bar{X}$ -outside-common.*

*Proof.* If  $\alpha < 2/3$ , then the statement follows from Lemmas 3 and 13. Let  $\alpha \geq 2/3$ . The cardinality  $N$  of the largest independent set in  $G(n, p)$  is given in Lemma 2. Moreover, by Hoeffding-Azuma inequality,  $\frac{N - \mathbb{E}N}{n^{1/2+\delta}} \xrightarrow{P} 0$  for any  $\delta > 0$ . In particular,  $\mathbb{E}N \sim 2(1 - \alpha)n^\alpha \ln n$ .

We construct a maximal independent set  $X$  by adding vertices step by step in the following way. At the first step, we put  $v_1 := 1$  into  $X$  and remove 1 and all neighbors of 1 from  $V$ . At the  $i$ -th step ( $i \geq 2$ ), we put  $v_i := \min\{j : j \in V\}$  into  $X$  and remove  $v_i$  and all its neighbors from  $V$ . At the final step, we get  $X = \{v_1, \dots, v_{|X|}\}$  and  $V = \emptyset$ .

Consider the event  $A$  that there is a vertex  $v$  outside  $X$  with  $N(v) \subset X$ . For  $i \in \{1, \dots, |X|\}$ , let  $A_i$  be the event “there is a neighbor of  $v_i$  such that all its neighbors are in  $X$ ”. We clearly have  $A = \cup A_i$ . For each step  $i$  and  $v \in N(v_i)$  all edges between  $v$  and  $X_i := X \setminus \{v_1, \dots, v_i\}$  appear mutually independently with probability  $p$ . For any  $c > 0$  and  $n$  large enough we have

$$\begin{aligned} \mathbb{P}(\exists u (u \sim v_i) \wedge (N(u) \subset X)) &\leq n\mathbb{P}(N(v) \subset X) \leq \\ n \left[ 1 - \mathbb{P} \left( |N(v) \cap X_i| < (c+1)p|X_i|, |N(v)| > \frac{1}{2}n^{1-\alpha}, |X_i| < 2n^\alpha \ln n \right) \right]. \end{aligned}$$

Let  $c = \frac{8}{1-\alpha}$ . By Chernoff bound,

$$\mathbb{P}(|N(v) \cap X_i| \geq (1+c)p|X_i|) \leq e^{-\frac{3}{1-\alpha}p|X_i|} \quad \text{and} \quad \mathbb{P} \left( |N(v)| \leq \frac{1}{2}n^{1-\alpha} \right) \leq e^{-\frac{1}{8}n^{1-\alpha}}.$$

By Hoeffding-Azuma inequality,

$$\begin{aligned} \mathbb{P}(|X| \geq 2n^\alpha \ln n) &\leq e^{-(4\alpha)^2(\ln n)^2 n^{2\alpha-1}(1+o(1))}, \\ \mathbb{P}(|X| < (1-\alpha)n^\alpha \ln n) &\leq e^{-(1-\alpha)^2(\ln n)^2 n^{2\alpha-1}(1+o(1))}. \end{aligned}$$

We have the following chain of inequalities:

$$\begin{aligned} 1 - \mathbb{P} \left( |N(v) \cap X| < 2p|X_i|, |N(v)| > \frac{1}{2}n^{1-\alpha}, |X_i| < 2n^\alpha \ln n \right) &\leq \\ \mathbb{P}(|X_i| \geq 2n^\alpha \ln n) + \mathbb{P} \left( |N(v)| \leq \frac{1}{2}n^{1-\alpha} \right) + \mathbb{P}(|X| < (1-\alpha)n^\alpha \ln n) &+ \\ \mathbb{P} \left( |N(v) \cap X_i| \geq \left( 1 + \frac{8}{1-\alpha} \right) p|X_i| \mid |X_i| \geq (1-\alpha)n^\alpha \ln n \right) &\mathbb{P}(|X_i| \geq (1-\alpha)n^\alpha \ln n) \leq \\ e^{-(4\alpha)^2(\ln n)^2 n^{2\alpha-1}(1+o(1))} + e^{-\frac{1}{8}n^{1-\alpha}} + e^{-(1-\alpha)^2(\ln n)^2 n^{2\alpha-1}(1+o(1))} &+ e^{-3 \ln n}, \end{aligned}$$

where the last inequality holds due to the independence of  $N(v)$  and  $X_i$ . Finally, we get

$$\mathbb{P}(A) \leq \sum_{i=1}^n \mathbb{P}(A_i) \leq n^2 e^{-3 \ln n(1+o(1))} \rightarrow 0.$$

□



We go on to the analysis of types 7 – 12. By Theorem 6 (note that it is also holds for induced subgraphs), a.a.s. there are  $v_1, v_2, v_3 \in V_n$  such that  $v_1 \sim v_2, v_2 \sim v_3, v_1 \not\sim v_3$ . Put  $X := \{x_1, x_3\}$ . By Lemma 10, a.a.s. the set  $\bar{X}$  is common. By Lemmas 1, 4, a.a.s.  $X$  has type 7.

Let  $s = \lfloor 1/\alpha \rfloor + 1$ . By Lemma 3, there exists an independent set  $X \subset V$  such that  $|X| = s$  and there are no common neighbors of vertices of  $X$  in  $G(n, p)$ . By Lemma 1, the set  $\bar{X}$  is common. By Lemmas 1, 4, a.a.s. the set  $X$  has type 8.

By Lemma 17, if  $\alpha \geq 1/3$ , then a.a.s. there is no sets of type 9 in  $G(n, p)$ . If  $\alpha < 1/3$ , then by Lemmas 13 and 17 a.a.s.  $G(n, p)$  contains a set of type 9.

The set  $X$  of type 10 exists by Lemmas 9 and 18.

If  $\alpha < 2/3$ , then a.a.s. in  $G(n, p)$  there are no sets of types 11, 12 by Lemma 13. If  $\alpha \geq 2/3$ , then, by Lemmas 13 and 14, in  $G(n, p)$  there is a set  $X$  of type 12. Obviously, a.a.s. it has a subset of type 11 as well.

### Types 13 – 26.

Let  $\alpha \geq 1/2$ . Applying Lemma 5 as in the proof of Lemma 13, we get that there are four vertices  $v_1, \dots, v_4$ , that form an independent set, and such that no two of them share a common neighbor. Split  $V_n \setminus (\cup_{i=1}^4 N(v_i))$  randomly into two almost equal parts  $V_1$  and  $V_2$ . By Lemma 1, a.a.s. any vertex  $v \in V_n \setminus \{v_1, v_2, v_3, v_4\}$  has both neighbors and non-neighbors in both  $V_1$  and  $V_2$ .

To prove the a.a.s. existence of type 13, put  $X := \{v_1\} \cup N(v_2) \cup \{v_3\} \cup N(v_3) \cup V_1$ . One can see that  $v_1$  is  $X$ -inside-isolated,  $v_3$  is  $X$ -outside-isolated,  $v_2$  is  $\bar{X}$ -inside-isolated, and  $v_4$  is  $\bar{X}$ -outside-isolated.

For type 14, put  $X := \{v_1, v_2, v_4\} \cup N(v_2) \cup N(v_3) \cup V_1$ . For type 15 put  $X := \{v_1, v_3\} \cup N(v_2) \cup N(v_4) \cup V_1$ . For type 16 put  $X := \{v_1, v_2\} \cup N(v_2) \cup V_1$ . For type 17 put  $X := \{v_1, v_2, v_3, v_4\} \cup N(v_2) \cup V_1$ . For type 22 put  $X := \{v_1\} \cup N(v_1) \cup V_1$ .

The types 18, 19 and 24 are even easier to obtain. Take two nonadjacent vertices  $u_1, u_2$ , and split the set  $V_n \setminus (N(u_1) \cup N(u_2))$  into two almost equal parts  $U_1, U_2$ . Once again, a.a.s., any other vertex has both neighbors and non-neighbors in both  $U_1$  and  $U_2$ .

To obtain type 18, put  $X := \{u_1, u_2\} \cup U_1$ . To obtain type 19, put  $X := \{u_1\} \cup U_1$ . To obtain type 24, put  $X := U_1$ .

To obtain type 23, simply split randomly  $V$  into two almost equal parts and choose  $X$  to be equal to one of these parts. Then a.a.s. all vertices are  $X$ -common and  $\bar{X}$ -common.

We are left to deal with types 20, 21, 25, 26. Let  $\alpha < 1/2$ . Take any vertex  $v$  and its neighbor  $w$ . A.a.s.,  $|N(w) \cap N(v)| = o(|N(v)|)$ . Put  $X := N(v) \setminus N(w)$ . Then  $X$  is sparse, it has a dominating vertex  $v$ , which is a.a.s. not special. Moreover, a.a.s.  $|X| = \Theta(n^{1-\alpha})$ , which

by Lemma 14 means that any vertex from  $\bar{X} \setminus \{v\}$  is  $\bar{X}$ -outside-common. By Lemma 10,  $X$  is of type 20.

To obtain a set of type 25, consider any vertex  $v$  and the set of its neighbors  $N(v)$ . Put  $X := N(v) \setminus \{w\}$ , where  $w$  is a neighbor of  $v$ . A.a.s.,  $\deg_X(v) > 0$  for each  $v \in V$  by Lemma 1. This implies that a.a.s. there is no  $X$ -inside-isolated vertex. Moreover, there are no  $\bar{X}$ -outside-isolated vertices by Lemma 14. Therefore,  $X$  is a.a.s. of type 25.

Let  $\alpha \in (0, 1)$ . By Theorem 6 and Lemma 4, the graph  $G(n, p)$  a.a.s. contains an induced subgraph, isomorphic to a triangle with a hanging edge:  $v_1, v_2, v_3, v \in V$ ,  $v_i \sim v_j$ ,  $v \sim v_1$ ,  $v \approx v_2$ ,  $v \approx v_3$ . Put  $X := \{v, v_2, v_3\}$ . Then  $X$  is sparse, and it has a dominating vertex  $v_1$ . Moreover, by Lemma 14, a.a.s. there are  $\bar{X}$ -outside-isolated vertices and there are no  $X$ -outside-isolated vertices. That is,  $X$  is a.a.s. of type 21.

Let  $X$  be of type 26 and let  $v$  be  $X$ -dominating. Then the subgraph, induced on  $X \cup \{v\}$ , has  $|X| + 1$  vertex and at least  $\lceil \frac{3}{2}|X| \rceil$  edges (each vertex of  $X$  has degree at least 1 in  $X$ ). Moreover,  $|X| > 3$ . It is not difficult to see that there is a subset  $X_0$  in  $X$  such that  $|X_0| = 4$  and the graph induced on  $X_0 \cup \{v\}$  contains two triangles sharing exactly one vertex. Thus,  $G(n, p)|_{X_0 \cup \{v\}}$  has density at least  $6/5$ , and by Theorem 6,  $G(n, p)$  a.a.s. does not contain such subgraphs for  $\alpha > 5/6$ .

If  $\alpha < 5/6$ , the subgraph (denote it by  $G|_X$ ) described in the previous subgraph do a.a.s. exist in  $G(n, p)$ . Then  $X$  is common, it has a dominating vertex  $v$ , and a.a.s. it has  $\bar{X}$ -outside-isolated vertices. Therefore,  $X$  is of type 26. Table 1 is verified completely.

### 4.3 Pairs of subsets for $\alpha > 1$

We may suppose that  $\alpha = 1 + \frac{1}{l}$  for  $l \geq 7, l \neq 8$ . Assume that Spoiler chooses a set  $X$ . Then Duplicator chooses a set of vertices  $Y \subset V(H)$ . He needs to get the pair  $Y, \bar{Y}$  of the same type as  $X, \bar{X}$  (see Section 4.1.2). Below, we describe the algorithm which Duplicator utilizes to achieve his goal. For  $Z \in \{X, \bar{X}\}$  put  $Y(Z) := \{Y, \text{ if } Z = X; \bar{Y}, \text{ if } Z = \bar{X}\}$ . The notation  $Z, Y(Z)$  is used for shorthand, and, in what follows, Duplicator makes each of his steps based on the information about *both* choices of  $Z$  *simultaneously*. Recall that we assume that  $X$  is nontrivial ( $|X| \geq 2$  and  $|\bar{X}| \geq 2$ ).

First, Duplicator checks, whether there are some isolated vertices in  $Z$ . Denote the set of isolated vertices in a graph  $W$  by  $I(W)$ . Duplicator splits  $I(H)$  arbitrarily according to the rule  $I(G) \cap Z \neq \emptyset$  iff  $I(H) \cap Y(Z) \neq \emptyset$ .

Let us call connected components of a graph with at least one edge *nonempty components*. Next, Duplicator verifies, whether there is a nonempty component  $C$  of  $G$  such that  $C \cap Z = \emptyset$ . The two answers to this question leads to two principally different cases.

**Case 1** Assume each component  $C$  of  $G$  intersects both  $X$  and  $\bar{X}$ . Then there are no dominating vertices neither for  $X$ , nor for  $\bar{X}$ . Moreover, since  $G(n, p)$  a.a.s. contains connected components that are edges, these edges in  $G$  are split between  $X$  and  $\bar{X}$ , and so both contain inside-isolated, outside-common vertices.

If neither  $X$  nor  $\bar{X}$  contains no inside-common vertices, then basically  $X, \bar{X}$  define a proper two-coloring of  $G$ . In this case Duplicator chooses any two-coloring of the nonempty components of  $H$  and puts  $Y$  to be one of these classes (together with the possibly chosen isolated vertices).

Denote a simple path on  $n$  vertices by  $P_n$ . Assume that  $Z$  contains an inside-common, outside-isolated vertex (which implies that  $Z$  has an inside-common, outside-common vertex as well). Then  $G$  contains  $P_3$ , which implies that  $\ell \geq 2$ , and so there are components of  $H$  that are isomorphic to  $P_3$ . In that case Duplicator takes such a path and puts two consecutive vertices into  $Y(Z)$ , while the third vertex into its complement.

If  $Z$  contains an inside-common, outside-common vertex, but no inside-common, outside-isolated vertex, then  $G$  contains  $P_4$ , which implies that  $\ell \geq 3$ , and so there are components of  $H$  isomorphic to  $P_4$ . In that case Duplicator takes such a path and puts two non-leaf vertices into  $Y(Z)$ , while the two leaf vertices into its complement.

After these two steps both  $Y$  and  $\bar{Y}$  have the same types of inside-common vertices as  $X$  and  $\bar{X}$ , respectively. The remaining all but at most two nonempty components of  $H$  are then properly colored in two colors and split between  $Y$  and  $\bar{Y}$ . It is not difficult to see that at the end the pair  $Y, \bar{Y}$  has the same type as  $X, \bar{X}$ .

**Case 2** Assume that there is a component  $C$  such that  $C \cap Z = \emptyset$ . We may w.l.o.g. assume that  $Z = X$ . This implies that  $\bar{X}$  contains inside-common, outside-isolated vertices (recall that we do not care about inside-isolated, outside-isolated vertices, since they are simply isolated vertices in  $G$  and were treated in the beginning of the algorithm). Therefore,  $\bar{X}$  does not belong to one connected component, and so does not have dominating vertices. This implies that the only ambiguity concerning the types of vertices in  $\bar{X}$  is what types of outside-common vertices it has, and what types of outside-dominating vertices it has.

In what follows we describe the strategy of Duplicator to construct step by step a forest  $T$  and a subset  $Y'$  of its vertices such that, first, for some union of components  $C$  in  $H$  the graph  $H|_C$  is isomorphic to  $T$ , and, second, for the  $Y \subset C$  that corresponds to  $Y'$  the pair  $Y, \bar{Y}$  is a pair of the same type as  $X, \bar{X}$ .

First, Duplicator checks the type of  $X$ :

- if  $X$  is complete, then Duplicator takes an edge;
- if  $X$  is dense, then Duplicator takes  $P_3$ ;
- if  $X$  is common, then Duplicator takes two disjoint edges;

- if  $X$  is independent, then Duplicator takes two isolated vertices.

The vertices produced on this step form a set  $Y'$ .

Next, Duplicator checks the types of vertices inside  $X$ :

- if all the vertices *of a given inside type* in  $X$  are outside-common, then he attaches a new edge to each vertex that has this type in  $Y'$ ;
- if some, but not all of the vertices of a given inside type in  $X$  are outside-common, then he attaches a new edge to *exactly one* vertex of  $Y'$  of this inside type;
- if there is a dominating vertex for  $X$  (which is possible only in the case if  $X$  is an independent set), then he adds a vertex and connects it to each of the vertices of  $Y'$ .

The vertices produced on this step form a set  $A_1$ .

Finally, Duplicator checks the outside types of vertices in  $\bar{X}$ :

- if there is an outside-dominating vertex in  $\bar{X}$ , then Duplicator attaches an edge to the corresponding vertex in  $A_1$  iff it is inside-common;
- if all the outside-common vertices in  $\bar{X}$  are inside-common, then he attaches a new edge to each vertex from  $A_1$ ;
- if some outside-common vertices in  $\bar{X}$  are inside-common and some are inside-isolated, then he attaches a new edge to one (no matter which one) vertex in  $A_1$ .

The vertices added at this step form a set  $A_2$ . In the last step we may run into a small trouble: if  $|A_1| = 1$ , then we cannot have both types of outside-common vertices in  $A_1$ . In this case we check the type of the vertex in  $Y'$  that is connected to the only vertex in  $A_1$  and connect a new edge to *the vertex  $v \in Y'$  of the same type and which lies in the smallest component of a graph induced on the set  $Y' \cup A_1 \cup A_2$* . The set added at this “trouble” step we call  $A_3$ . Note that  $|A_3| \leq 1$ . Note that if  $|A_1| = 0$ , then there are no outside-common vertices in  $\bar{X}$ .

Put  $T$  be the described forest on  $Y' \cup A_1 \cup A_2 \cup A_3$ . It is clear by the definition of  $T$  that, if a set of connected components  $C$  in  $H$  is isomorphic to  $T$  and  $Y$  is a subset of  $V(H)$  that corresponds to  $Y'$ , then the pair  $Y, \bar{Y}$  has the same type as  $X, \bar{X}$ . Note that at each step we increased the size of the components by the least possible value. This means that in  $G$  there is a connected component  $F$  such that  $v(F) \geq v(T')$  where  $T'$  is the largest component of  $T$ . So,  $\ell \geq v(T') - 1$ . Lastly, it is not difficult to check that  $v(T')$  is either at most 7, or equal to 9. Therefore, by properties T2, T3, the set  $Y$  exists.

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